Monte Carlo renormalization-group approach to the Bak-Sneppen model

Bernhard Mikeska*
II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany
(Received 16 October 1996)

A recent renormalization-group (RG) approach to a modified Bak-Sneppen model is discussed. We propose a self-consistency condition for the blocking scheme to be essential for a successful RG method applied to self-organized criticality. A method realizing the RG approach to the Bak-Sneppen model is presented. It is based on the Monte Carlo importance sampling idea. This technique performs much faster than the original proposal. Using this technique we cross-check and improve previous results. [S1063-651X(97)01603-6]

PACS number(s): 64.60.Lx, 64.60.Ak, 05.40.+j, 05.70.Jk

Biological evolution has been a candidate for self-organized criticality (SOC) [1] for a long time. The Bak-Sneppen (BS) model [2] is a model describing biological evolution as a self-organized critical phenomenon. A renormalization-group (RG) approach to calculating critical exponents of the BS model was presented in [3]. This paper presents a Monte Carlo technique to realize the approach of [3] and provides thereby an easy way of cross-checking the RG method. It accelerates and simplifies the calculation by using a Monte Carlo inspired technique. We are able to improve the results given in [3]. The paper is organized as follows. We redefine the BS model and review the main ideas of the RG approach. We discuss the main tool of this approach, the run-time-statistics (RTS) technique, which was presented in [4]. Also we introduce our technique to perform the RG approach more efficiently. The fixed point properties are given and basic critical exponents are calculated.

Following Ref. [3] we modify the BS model and define a *left-or-right* (L/R) BS model: Consider N real variables $\boldsymbol{\phi}^{(t)}=(\phi_i^{(t)})$, $i\in\{1,\ldots,N\}$ with values $0.0\leqslant\phi_i^{(t)}<1.0$. At each time t determine index $i(t)$ indicating the smallest value $\phi_{i(t)}$ among all ϕ_j. Replace the value $\phi_{i(t)}$ with a new value x with probability $a(x)$, $\int a(x)dx=1$. Choose with equal probability the left or right neighbor of the active site $i(t)$ and replace it with a new value y with probability $b(y)$, $\int b(y)dy=1$ [5]. After equilibration almost all variables ϕ_j have a higher value than some ρ_c. A ρ avalanche is defined to start at time t_i with the minimal value $\phi_{i(t_i)}=\rho$, lasts as long as consecutive minimal values are smaller than ρ, and stops at t_f if $\phi_{i(t_f)}\geqslant\rho$. The temporal extension is given by $s\equiv t_f-t_i$. The spatial extension ξ is defined as the largest extent of active sites involved, $\xi\equiv\max_{t_i\leqslant t_1,t_2\leqslant t_f}|i(t_1)-i(t_2)|$. Avalanches keep themselves running by generating variables smaller than ρ. If this number of variables smaller than ρ, $n_t(\rho)\equiv\Sigma_i\Theta(\rho-\phi_i^{(t)})$, decreases with time during a ρ avalanche the avalanche is called subcritical and will die out. On the other hand, if $n_t(\rho)$ increases, the avalanche is called supercritical and will last forever. If $n_t(\rho)$ stays constant we observe a critical avalanche. Critical avalanches obey power laws for their spatial and temporal size, $P_\xi(\xi)\sim\xi^{-\tau_\xi}$ and $P(s)\sim s^{-\tau}$, respectively. The connection between spatial and temporal size is given by $s\sim\xi^z$. The critical exponents τ_ξ, τ, and z are connected via the scaling relation $z\tau=\tau_\xi-1+z$. Extensive numerical studies have been performed to calculate these exponents for the original BS model [6–8].

The goal of a RG approach is threefold: calculate critical exponents in an independent analytical or semianalytical way, prove the attractive nature of the critical state, and demonstrate the concept of universality. The RG approach [3] considers small avalanches as objects to be integrated out and provides a mapping of larger avalanches onto smaller ones. This goes in parallel with a modification of the dynamical rules at coarser scales.

To be more precise, let us denote the fine scale with index (l), the next coarser scale with $(l+1)$. Block variables are Φ_I, fine variables are ϕ_i. Using a block factor of 2, the simplest block transformation conserving the spatial avalanche structure is

$$\Phi_I^{(t')}=\min\{\phi_{2i}^{(t)},\phi_{2i+1}^{(t)}\}. \tag{1}$$

Then the dynamics on the coarser level is again based on the selection of the minimum. To perform one update on the coarse grid, i.e., updating $\Phi_{I(t)}$ and either the left or right neighbor $I(t)\pm1$, we have to consider a process with four neighboring variables on the finer scale being updated. Thus a dynamical blocking of time occurs and we have different time scales on the fine and coarse level. The mapping connecting both time scales is dynamic, i.e., it depends on the stochasticity of the temporal evolution. This will provide us with a modified dynamics based on new distributions $a^{(l+1)}(x)$ and $b^{(l+1)}(x)$. The initial and final states of one updating step on the coarse level are obtained by applying Eq. (1) to the initial and final configuration of the corresponding fine level process. The spatial and temporal blocking of the dynamical process is schematically shown in Fig. 1. The block transformation and the modified dynamics have to be designed such that it makes statistically no difference for the final state $\Phi^{(1)}$ if we apply first T fine updates and then the block transformation or if we apply first the block transformation and then perform one coarse update using the coarse dynamics: the blocking diagram has to be commutative for a successful RG approach.

*Address after June 1, 1997: Landestheater Tübingen, Eberhardstr. 6, 72072 Tübingen, Germany.

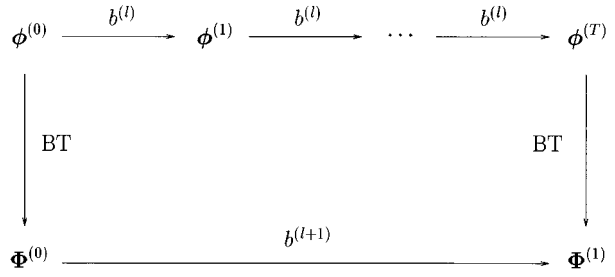

FIG. 1. Concept of blocking in space and time using a block transformation BT. T updates on the fine level l are performed until four neighboring variables are changed. The result of this process defines a contribution to one time step on the coarse level $l+1$.

Assume that we already know the probability distributions $\tilde{a}^{(l+1)}(x)$ and $\tilde{b}^{(l+1)}(x)$ of variables $\Phi_{I(t)}$ and $\Phi_{I(t)\pm 1}$ in the final state on the coarse level. It is then possible [3] to perform a variable transformation leading to a uniform distribution $a^{(l+1)}(y)=\Theta(y)\Theta(1-y)$ and a new distribution $b^{(l+1)}(y)$. This reduces the possible RG proliferation to the distribution b. All other rules (search for the minimum, choose left or right neighbor, replace the value of the minimal variable with a uniformly distributed one) are invariant under the RG transformation.

Scale invariant behavior is expected only for avalanches that are critical, i.e., $\phi^{(l)}_{i(t_i)}=\rho^{(l)}_c$. Therefore one has to consider only critical processes for the mapping of a sequence of fine updates onto one coarse update. Only processes α with four neighboring sites being updated after T_α time steps are relevant. Processes that stop earlier because $\phi^{(l)}_{i(t')}\geqslant\rho^{(l)}_c$ for $t'>t_i$ do not contribute since they represent small fluctuations with avalanche size $S\leqslant 3$ and are neglected in the RG step. The parameter $\rho^{(l)}_c$ is determined from the condition that $n_t(\rho^{(l)}_c, b^{(l)})$ should stay constant for critical avalanches, see Eq. (9) below.

We now want to discuss the run-time-statistics approach developed in [4]. Denote the probability distribution of ϕ_i at time t by $p_{i,t}(x)$. The conditional probability that the minimal variable is located at site i is then

$$p^t_{\min,i}(x)=p_{i,t}(x)\prod_{j\neq i}\int_x^1 p_{j,t}(y)dy. \tag{2}$$

For the blocking procedure we are interested in the probability that $\phi^{(t)}_i$ is the minimal site *and* that the avalanche does not stop, i.e., $\phi^{(t)}_i<\rho$, which is

$$\mu_{i,t}(\rho)=\int_0^\rho p^t_{\min,i}(x)dx. \tag{3}$$

Exploiting the information about the position of the minimal value ϕ_i, $\phi_i<\rho$, the probability distribution $p_{j,t}(x)$ of all other sites j modifies in the next time step to

$$p_{j,t+1}(x)=\mathcal{N}p_{j,t}(x)\int_0^\rho p_{i,t}(y)\Theta(x-y)\times\prod_{k\neq i,j}\int_y^1 p_{k,t}(z)dzdy. \tag{4}$$

$\mathcal{N}$ is a normalization factor. The probability distribution of the minimal site $i(t)$ and its left or right neighbor change according to the rules of the L/R BS model to

$$p_{i(t),t+1}(x)=\Theta(x)\Theta(1-x), \tag{5a}$$

$$p_{i(t)\pm 1,t+1}(x)=b(x). \tag{5b}$$

Note that Eq. (4) is different [9] from the corresponding equations in [3,4]. This leads to different results of the fixed point properties, see below. Using the iterative rules (4) and (5) we can calculate the weight ν_α of a given process α contributing to the blocking procedure,

$$\nu_\alpha(\rho)=\frac{1}{2^{T_\alpha}}\prod_{t=1}^{T_\alpha}\mu_{i_\alpha(t),t}(\rho). \tag{6}$$

The final probability distributions in the four updated variables contribute to the blocked distributions $\tilde{a}$ and $\tilde{b}$ at the next coarser level according to the weight of process α. Using the variable transformation described in [3], $\tilde{a}(x)$ may be rescaled to a uniform distribution in the interval $[0,1]$ and the next renormalization procedure can be iterated.

To realize the renormalization-group approach using the RTS technique, one has to calculate the weights and final probability distributions of all contributing processes for each iteration during the computation of ρ_c, using Eq. (9) below. Then one has to perform the same task for all paths using the value ρ_c. Finally the block transformation (1) has to be applied. However, there is a huge amount of contributing processes: up to a length of $T_\alpha\leqslant 20$ we estimated a total number of $O(10^{12})$ contributing processes. It is also clear that most of them, especially of the longer processes (e.g., the ones with the minimal site at the same place for a long time) are of very small probability. This observation calls to mind the problems in ''simple sampling'' Monte Carlo algorithms, calculating properties and probabilities of all possible states in phase space. It is more convenient to use an ''importance sampling'' method generating the contributing processes α according to their weight ν_α. Then all generated processes of the ensemble contribute with equal probability. In other words, we evaluate RTS integrals using the Monte Carlo importance sampling method.

To generate a single relevant process α using distribution $b^{(l)}(x)$, we start with a variable $\phi_{i_\alpha(0)}=\rho$, and apply the rules of the L/R BS model until four neighboring sites are updated. For this process we count the number of variables smaller than ρ after the first time step as well as in the final state,

$$n_1^{(\alpha)}=\sum_i\Theta(\rho-\phi_i^{(t=1)}), \tag{7a}$$

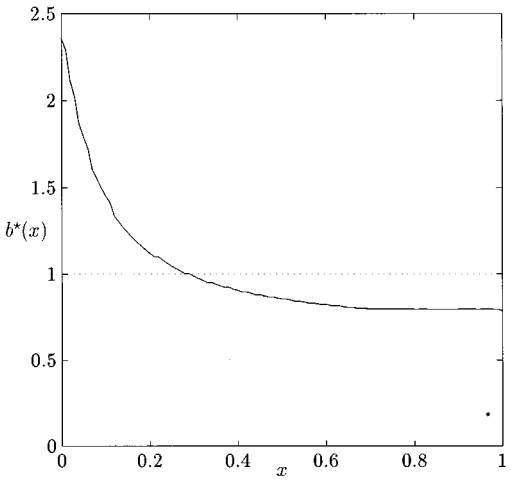

FIG. 2. The fixed point distribution $b^{\star}(x)$ is observed already after one renormalization step. For each RG step 10^9 relevant processes have been generated.

$$n_{T_\alpha}^{(\alpha)} = \sum_i \Theta(\rho - \phi_i^{(t=T_\alpha)}). \tag{7b}$$

The sum is over all updated variables of process α. If $n_1^{(\alpha)}=0$ or $n_{T_\alpha}^{(\alpha)}=0$, the generated process is not a relevant process, since it stopped, i.e., $\phi_{i(t)}>\rho$, before four neighboring sites are updated. Using this notation we can easily write down $n_{t=1}(\rho,b^{(l)})$ and $n_{t=t_{\text{final}}}(\rho,b^{(l)})$ obtained from the generation of an ensemble of N processes [10]

$$n_{t=1}(\rho,b^{(l)}) = \frac{1}{N}\sum_\alpha n_1^{(\alpha)}, \tag{8a}$$

$$n_{t=t_{\text{final}}}(\rho,b^{(l)}) = \frac{1}{N}\sum_\alpha n_{T_\alpha}^{(\alpha)}. \tag{8b}$$

Since we are interested only in critical avalanches we have $n_t(\rho_c^{(l)},b^{(l)})=\text{const}$ and obtain $\rho_c^{(l)}$ as a solution of

$$n_1(\rho_c^{(l)},b^{(l)}) = n_{t_{\text{final}}}(\rho_c^{(l)},b^{(l)}). \tag{9}$$

Using the secant method it converges in about five iterations. Once we know the critical $\rho_c^{(l)}$ at level l for given $b^{(l)}$, we are able to evaluate all *critical* relevant processes α starting with $\phi_{i_\alpha(0)}=\rho_c^{(l)}$ for their contribution to the blocked distributions $\tilde{a}^{(l+1)}(x)$ and $\tilde{b}^{(l+1)}(x)$. A relevant process α contributes with

$$\tilde{a}_\alpha^{(l+1)}(x) = \delta(x-m_\alpha), \quad m_\alpha = \min\{\phi_{2j}^{(t_\alpha)},\phi_{2j+1}^{(t_\alpha)}\}, \tag{10a}$$

$$\tilde{b}_\alpha^{(l+1)}(x) = \delta(x-n_\alpha), \quad n_\alpha = \min\{\phi_{2k}^{(t_\alpha)},\phi_{2k+1}^{(t_\alpha)}\}. \tag{10b}$$

The first active site $i_{\alpha(t=0)}$ has index $2j$ or $2j+1$ and the pair $(2k,2k+1)$ is the right or left neighbor of $(2j,2j+1)$. The four indices $2j,2j+1,2k,2k+1$ have been updated in process α. For an ensemble of N' relevant processes α, each occurring with a probability $\nu_\alpha(\rho_c^{(l)})$, we obtain

$$\tilde{a}^{(l+1)}(x) = \frac{1}{N'}\sum_\alpha \tilde{a}_\alpha^{(l+1)}(x), \tag{11a}$$

$$\tilde{b}^{(l+1)}(x) = \frac{1}{N'}\sum_\alpha \tilde{b}_\alpha^{(l+1)}(x). \tag{11b}$$

To these preliminary distributions we apply the variable transformation described in [3], thereby switching back to a uniform distribution $a^{(l+1)}(x)$ and a transformed distribution $b^{(l+1)}(x)$. Now the described procedure may be iterated.

An ensemble of processes α, each occurring with probability ν_α, is generated by applying the rules of the L/R BS model to processes starting with $\phi_{i_\alpha(0)}=\rho$ as long as four neighboring sites have not been updated. Once this happens, we stop the process and save its initial and final state. We reject nonrelevant processes. Then the process α occurs automatically with its correct weight ν_α without the need of an explicit calculation of ν_α using RTS. Then we start the same procedure again for the next process that is generated completely independently from the previous one.

Being considerably simpler our method is able to reproduce the results of [3] with good accuracy in about one hour of CPU time on a workstation. Moreover, there is no need to extrapolate the results from $T_{\max}=20$ to $T_{\max}=\infty$ as done in [3]. In our approach we effectively consider $T_{\max}=\infty$, since we allow arbitrarily long relevant processes.

On the other hand, since we generate a finite number N of processes, we observe statistical fluctuations in the resulting distribution $b^{(l)}(x)$, whereas the method [3] gives in this sense an exact result (neglecting small errors due to the numerical integration routines). But this is a relatively small drawback compared to the advantage of speeding up the method, thereby having an effective method at hand to cross-check previous results.

Starting with a distribution $b^{(0)}(x)=\Theta(x)\Theta(1-x)$ at the finest level $l=0$, we observe that the distribution $b^{(l)}(x)$ converges very fast. It reaches its fixed point shape already after one renormalization step, see Fig. 2. Correspondingly the value of $\rho_c^{(l)}$ converges also very fast to its fixed point value $\rho_c^{\star}=0.5954$. For the calculation of $b^{(l)}(x)$ we evaluated 10^9 relevant processes at each renormalization step. For the determination of $\rho_c^{(l)}$ we evaluated 10^7 processes per it-

TABLE I. Comparison of results for critical exponents using the renormalization-group approach and simulations. The renormalization-group approach applies to the L/R version of the BS model. RTS approach results are given in [3].

	RG method		Simulation
	MC approach	RTS approach	Original BS
τ	1.1246(1)[a]	1.1204[a]	1.08(1)
τ_ξ	1.2911(1)	1.2766	
z	2.336(1)	2.2975	2.43(1)[b]

[a]These results have been obtained from the scaling relation.
[b]This result is given in [7].

eration of the secant method and renormalization step. The results of [3] differ significantly, which is probably due to the error in one of the RTS-iteration equations [9].

The probability distribution $P^\star(T)$ for the length T of relevant processes, corresponding to the fixed point properties $b^\star$ and $\rho_c^\star$ decays exponentially: $P^\star(T) \sim e^{-\text{const} \times T}$. The minimal length of a relevant process is $T=3$. At $T=30$ the probability is of order 10^{-5}.

To compute the critical exponents from the fixed point properties we follow the discussion of [3] and use Eqs. (5) and (6) given therein. For details the reader is referred to [3]. From the probability distribution $P^\star(T)$ we obtain the exponent $z=2.336(1)$. For the calculation of exponent τ_ξ, describing the spatial scaling of avalanche sizes, the probability $K^\star$, that the site with minimal value is always the same (until the avalanche stops), has to be known. The value of $K^\star$ may be obtained again using our Monte Carlo approach. We have to count the fraction of processes with the minimum at the same site at each time step until they stop due to the condition $\phi_{i_\beta(T_\beta)} \geqslant \rho_c^\star$. Using the fixed point distribution $b^\star(x)$, we find $K^\star=0.1827(1)$, from which follows $\tau_\xi=1.2911(1)$. Using the scaling relation, connecting τ_ξ, τ, and z, and our result for the dynamical exponent z we obtain for $\tau=1.1246(1)$.

The values of exponents obtained via the renormalization-group approach and via numerical simulation are compared in Table I. Due to the probabilistic nature of our approach the fixed point distribution fluctuates a little. The relative error in $\rho_c^\star$ z, $K^\star$, τ_ξ, and τ induced by this fluctuation has been estimated to be of order of 10^{-3}. The results obtained by numerical integration of the RTS equations given in [3] differ significantly due to the error in one of the RTS equations. The results of the Monte Carlo RG approach show a coincidence up to 4% with numerical values for the exponents of the original BS model. It is assumed that the L/R version is in the same universality class as the original BS model.

In conclusion we have introduced a Monte Carlo renormalization-group method for the L/R Bak-Sneppen model. Previous results for basic critical exponents are improved. We have proposed a self-consistency condition for the blocking diagram in RG methods applied to SOC systems.

I would like to thank G. Mack and Y. Xylander for stimulating discussions. This work was supported in part by the Deutsche Forschungsgemeinschaft and the Studienstiftung des deutschen Volkes.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993).
[3] M. Marsili, Europhys. Lett. **28**, 385 (1994).
[4] M. Marsili, J. Stat. Phys. **77**, 733 (1994).
[5] In the case of $a(x) \equiv b(x) \equiv \Theta(1-x)\Theta(x)$ and replacing both the right and left neighbor of the active site this definition reduces to the original BS model [2].
[6] P. Grassberger, Phys. Lett. A **200**, 277 (1995).
[7] M. Paczuski, S. Maslov, and P. Bak, Phys. Rev. E **53**, 414 (1996).
[8] B. Mikeska, Ph.D. thesis, Universität Hamburg, 1996 (obtainable from author: mikeska@x4u2.desy.de).
[9] Marsili did not take into account the fact that $m_{i,t}(x)$ and $p_{j,t}(y)$, $j \neq i$ (using the notation of [3]) are not mutually independent and it is not allowed to multiply them to obtain the new conditional probability. See also A. Gabrielli *et al.*, J. Stat. Phys. **84**, 889 (1996).
[10] The ensemble of N processes includes also the nonrelevant ones.